Journal of Applied Mechanics and Technical Physics, Vol. 36, No. 2, 1995

PRESSURE ATTENUATION CURVE IN THE RELAXATION THEORY OF FILTRATION. CASE OF MULTIPLE RELAXATION TIMES

O. Yu. Dinariev

UDC 532.546

Here we derive an asymptotic expression for the pressure attenuation curve for the relaxation theory of percolation [flow through a porous medium], in which there is a family of purely dissipative internal relaxation processes when a saturated fluid flows through rock. The asymptote has been derived for the initial part of the pressure attenuation curve [1]; here we derive the curve for longer times – times on the same order as the internal relaxation times.

We recall the basic theory of isothermal relaxation flow in an undeformed homogeneous isotropic collector [2–6].

In the relaxation theory of percolation, Darcy's law

$$u^i(t, x^j) = -k\mu^{-1} \frac{\partial G}{\partial x^i}(t, x^j), \quad G = p + \rho\varphi \tag{1}$$

is replaced by the relaxation equation

$$u^i(t_0, x^j) = -k\mu^{-1} \int_{-\infty}^{+\infty} K(t_0 - t) \frac{\partial G}{\partial x^i}(t, x^j)\, dt. \tag{2}$$

In Eqs. (1) and (2), u^i is the percolation velocity, k is the permeability, μ is the shear viscosity of the fluid, p is the pressure, ρ is the density, and φ is the gravitational potential.

The kernel $K = K(t)$ characterizes the internal relaxation processes in system of the porous medium plus the saturated fluid. It is subject to a series of conditions which follow from physical and thermodynamic considerations.

1. $K(t)$ is a non-negative, monotonically decreasing function with dimensions of $(\text{time})^{-1}$.

2. $\int\limits_{-\infty}^{+\infty} /K(t) = 1$ is the condition for reducing (2) to (1) for slow processes.

3. $K(t) = 0$ for $t < 0$ (causality); $K(0) < +\infty$ is the condition for a finite signal velocity [7].

For an arbitrary function of time $f = f(t)$, we use $f_F = f_F(\omega)$ to denote its Fourier transform:

$$f_F(\omega) = \int_{-\infty}^{+\infty} e^{-i\omega t} f(t)\, dt.$$

According to the Payley–Wiener theorem [8], condition 3 makes the function $K_F = K_F(\omega)$ holomorphous in the lower half of the complex plane. It has been shown [6,7] that there is a dissipation condition

4. Re $K_F(\omega) > 0$ for $\mathrm{Im}(\omega) \leq 0$.

From condition 2 it follows that

$$K_F(0) = 1. \tag{3}$$

From condition 3 it follows that the asymptote

$$K_F(\omega) = k_1(i\omega)^{-1} + o(|\omega|^{-1}), \quad k_1 = K(0). \tag{4}$$

is valid in the holomorphous region.

Institute of Earth Physics, Russian Academy of Sciences, 123810 Moscow. Translated from Prikladnaya Mekhanika i Tekhnicheskaya Fizika, No. 2, pp. 119-125, March-April, 1995. Original article submitted May 10, 1994.

0021-8944/95/3602-0252$12.50

Because the kernel K is real,

$$\overline{K_F(\omega)} = K_F(-\bar{\omega}), \quad \operatorname{Im}\omega \leq 0. \tag{5}$$

Now we examine a specific functional form of the relaxation kernel in more detail. Because K_F is holomorphous in the lower half of the complex plane, the function form of the kernel is determined by singularities in the upper half of the complex plane. The contribution from "pole" singularities has the form $A_0/(\omega - \omega_0)$.

We now assume that the porous medium–saturated fluid system has a discrete spectrum of purely dissipative internal relaxation processes. Then the relaxation kernel can be represented as

$$K(t) = \sum_{n=1}^{N} A_n \tau_n^{-1} \exp(-t/\tau_n), \tag{6}$$

where the τ_n are the corresponding relaxation times ($0 < \tau_{n+1} < \tau_n$), and the $A_n > 0$ are the weighting factors. The Fourier transformation of (6) takes the form

$$K_F(\omega) = \sum_{n=1}^{N} A_n (1 + i\omega\tau_n)^{-1}. \tag{7}$$

It is easy to verify that conditions 1, 3, and 4 are fulfilled. From (3) it follows that

$$1 = K_F(0) = \sum_{n=1}^{N} A_n. \tag{8}$$

We now examine pressure attenuation in cylindrical symmetry and limit ourselves to the linear approximation. For the equation of state we use

$$p = p_0 + E(\rho - \rho_0)\rho_0^{-1}, \tag{9}$$

where E is the elastic bulk modulus of the fluid.

Mass must be conserved locally as the fluid flows through the porous medium:

$$\frac{\partial}{\partial t}(m\rho) + \frac{\partial}{\partial x^i}(\rho u^i) = 0, \tag{10}$$

where m is the porosity. By substituting Eqs. (2) and (9) into Eq. (10) and linearizing the result, and by considering the cylindrical symmetry, we obtain the integrodifferential equation

$$\frac{\partial}{\partial t} p(t_0, r) = \text{æ} \int_{-\infty}^{+\infty} K(t_0 - t) \Delta p(t, r)\, dt. \tag{11}$$

Here $\text{æ} = kE/(m\mu)$; r is the distance from the well axis; and $\Delta = \dfrac{\partial^2}{\partial r^2} + r^{-1} \dfrac{\partial}{\partial r}$ is the Laplacian operator. The parameter r varies in the range $r_1 \leq r \leq r_2$, where r_1 is the well radius and r_2 is the recharge radius. If the well operates with a yield $q = q(t)$ per unit thickness of the productive bed, then we have the boundary condition

$$q(t) = \lambda \int_{-\infty}^{+\infty} K(t_0 - t) \frac{\partial}{\partial r} p(t, r_1)\, dt, \quad \lambda = 2\pi r_1 \rho_0 \mu^{-1}. \tag{12}$$

In addition, there must be a boundary condition on the recharge boundary

$$p(t, r_2) = p_{\text{bed}}. \tag{13}$$

The problem (11)–(13) is linear; therefore it can be solved by the Fourier–Laplace transform method. In order to simplify the resultant equations, we will hereafter use a system of units in which

$$æ = r_1 = 1. \tag{14}$$

Because æ has units of $(\text{length})^2/\text{time}$, Eq. (14) hereafter fixes the units of length and time.

We now introduce a new unknown function

$$P = P(t,r) = p(t,r) - p_{\text{bed}}.$$

By taking the Fourier transform of (11)–(13), we obtain the differential equation

$$(\Delta - \alpha^2)P_F = 0 \tag{15}$$

with the boundary conditions

$$q_F \doteq \lambda K_F \frac{\partial P_F}{\partial r}\Big|_{r=1}, \quad \text{and} \quad P_F\Big|_{r=r_2} = 0. \tag{16}$$

In Eq. (15), $\alpha = \alpha(\omega)$ is a complex function determined by the relationships

$$\alpha^2 = i\omega/K_F(\omega), \quad \text{and} \quad \operatorname{Re}\alpha \geq 0. \tag{17}$$

We now show that Eqs. (17) determine a holomorphous function $\alpha = \alpha(\omega)$ for $\operatorname{Im}\omega < 0$ which is continuous all the way to the real axis.

In truth, the function $\alpha = \alpha(\omega)$ can become discontinuous only at points ω at which $\operatorname{Re}\alpha = 0$ and $\operatorname{Im}\omega \neq 0$; i.e., where

$$\operatorname{Re}\alpha^2 < 0, \quad \text{and} \quad \operatorname{Im}\alpha^2 = 0. \tag{18}$$

We note that

$$\operatorname{Im}\alpha^2 = \frac{\operatorname{Re}\omega \operatorname{Re} K_F + \operatorname{Im}\omega \operatorname{Im} K_F}{|K_F|^2} \tag{19}$$

and

$$\operatorname{Im} K_F(\omega) = -\int_0^{+\infty} e^{t \operatorname{Im}\omega} \sin(t \operatorname{Re}\omega) K(t)\, dt. \tag{20}$$

Because the kernel is monotonic, it follows from (20) that $(\operatorname{Re}\omega)(\operatorname{Im} K_F) \leq 0$. Now it follows from Eq. (19) and condition 4 that $\operatorname{Im}\alpha^2$ has the same sign as $\operatorname{Re}\omega$. If $\operatorname{Re}\omega = 0$, then $K_F = \operatorname{Re} K_F$ and $\alpha^2 > 0$. Therefore the condition (18) is impossible at all points in the lower half of the complex plane.

Thus, it has been proven that $\alpha(\omega)$ is holomorphous in the lower half of the complex plane. Because K_F is holomorphous and can be continued into the upper half of the complex plane [see (7)], then the function $\alpha(\omega)$ can be also. Here it will have singularities related to poles and zeroes in the function $K_F(\omega)$ as well as to the procedure of extracting the root in (17).

Now we examine a more specific function $F(s) = K_F(is)$ for $s > 0$. According to (5) this function takes on real values. Analysis of Eq. (7) shows that $F(s)$ increases monotonically in each of the intervals $0 < s < y_1$, $y_1 < s < y_2$, ..., $y_{N-1} < s < y_N$, where $y_i = \tau_i^{-1}$, and tends to $\pm\infty$ as s approaches y_i. Thus, points $s_1, \ldots, s_{N-1}$ exist, where this function becomes zero, so that $y_i < s_i < y_{i+1}$. We define $s_0 = 0$. Because the function $\alpha(\omega)$ is holomorphous, it can be continued over the whole complex plane, except for cuts along intervals

$$D_i: \quad s_i \leq \operatorname{Im}\omega \leq y_{i+1}, \quad \operatorname{Re}\omega = 0 \quad (i = 0, \ldots, N-1).$$

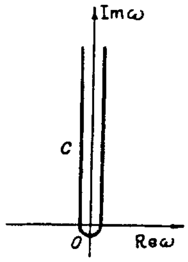

Fig. 1

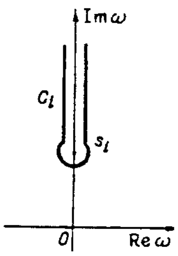

Fig. 2

Equation (15) has a general solution in terms of MacDonald functions [9]

$$P_F = C_1 K_0(\alpha r) + C_2 I_0(\alpha r),$$

where the constants C_1 and C_2 are determined from the boundary conditions (16):

$$P_F = \frac{q_F(-I_0(\alpha r_2)K_0(\alpha r) + K_0(\alpha r_2)I_0(\alpha r))}{\lambda K_F \alpha(K_0(\alpha r_2)I_1(\alpha) + K_1(\alpha)I_0(\alpha r))}. \tag{21}$$

If the well operates with a constant yield $q(t) = Q$, it is easy to show that $P = P(r) = \lambda^{-1}Q\ln(r/r_2)$. For the pressure-decay problem, we must set $q(t) = Q\theta(-t)$, where $\theta(t)$ is the Heavyside function.

We define

$$\Phi = P - \lambda^{-1}Q\ln(r/r_2).$$

Then Φ_F can be calculated from Eq. (21) if we set $q_F = Qi/(\omega - i\varepsilon)$, which corresponds to $q(t) = -Q\theta(t)$. Here ε is a small positive quantity which is set to zero in the final result.

Now we calculate the intermediate asymptote of the pressure attenuation curve, when r_2 approaches infinity in Eq. (21). By using the asymptotic values of the MacDonald functions [9], we find

$$\Phi_F = \frac{-q_F K_0(\alpha r)}{\lambda K_F \alpha K_1(\alpha)}.$$

Then the problem of determining the pressure attenuation curve reduces to calculating the function $\varphi(t) = \Phi\mid_{r=1}$. After we take the inverse transform, we obtain this function in the form

$$\varphi(t) = -(2\pi\lambda)^{-1}Q\, i \int_{-\infty}^{+\infty} (\omega - i\varepsilon)^{-1} f_1(\omega)e^{i\omega t}d\omega,$$
$$f_1(\omega) = \frac{K_0(\alpha)}{K_F \alpha K_1(\alpha)}. \tag{22}$$

We recall the expressions for the MacDonald functions [9]:

$$K_0(z) = -J(z^2)\ln(z/2) + W(z^2), \quad \text{and} \quad K_1(z) = z^{-1}(A(z^2)\ln z + B(z^2)). \tag{23}$$

Here $J(z)$, $A(z)$, and $B(z)$ are complete functions; $J(0) = B(0) = 1$; $A(0) = 0$; and $W(0) = -C$, where C is Euler's constant. The functions $K_0(z)$ and $K_1(z)$ have an infinite number of branches at $z = 0$. We will exclude these branches with a cut along the ray $\mathrm{Re}\, z < 0$, $\mathrm{Im}\, z = 0$.

In order to integrate (22), we deform the integration contour in the upper half of the complex plane to avoid the singularities in the integrand. All singularities (poles and cuts) occur on the semiaxis $\mathrm{Im}\,\omega > 0$, $\mathrm{Re}\,\omega = 0$; therefore the integral (22) transforms to an integral along the contour C (Fig. 1):

$$\varphi(t) = -(2\pi\lambda)^{-1} Q\, i \int_C \omega^{-1} f_1(\omega)\, e^{i\omega t}\, d\omega. \tag{24}$$

We now look for the asymptotic pressure attenuation curve for times large compared to the relaxation times τ_i. Formally this can be done by setting $t = \delta^{-1} t_*$ and $K(t) = K_*(\delta \cdot t)$, where δ is a small parameter and t_* and K_* are fixed, and by substituting these expressions into Eq. (24). Then we omit those terms in (24) which vanish as δ tends to zero and derive a formula from (24) for $\varphi(t)$ by replacing $f_1(\omega)$ by the function

$$f_2(\omega) = \frac{-\ln\alpha + \beta}{K_F}, \quad \beta = \ln 2 - C. \tag{25}$$

Now we note that

$$(i\omega K_F(\omega))^{-1} = \sum_{n=0}^{N-1} a_n (i\omega + s_n)^{-1} + b_0; \tag{26}$$

$$\frac{i\omega}{K_F(\omega)} = A \prod_{n=0}^{N-1} (i\omega + s_n) \prod_{n=1}^{N} (i\omega + y_n)^{-1}, \tag{27}$$

where a_i, $b_0 > 0$; $b_0 = k_1{}^{-1}$; $a_0 = 1$; and $A = \prod_{n=1}^{N-1} s_i{}^{-1} \prod_{n=1}^{N} y_i$. Substitution of Eqs. (26) and (27) into (25) and (24) leads to the formula

$$\varphi(t) = (2\pi\lambda)^{-1} Q \int_C f_3(\omega)\, e^{i\omega t}\, d\omega,$$
$$f_3(\omega) = \Big(\sum_{n=0}^{N-1} a_n (i\omega + s_n)^{-1} + b_0\Big)\Big(-\frac{1}{2}\sum_{n=0}^{N-1} \ln(i\omega + s_n) + \cdot\frac{1}{2}\sum_{n=1}^{N} \ln(i\omega + y_n) + \gamma\Big), \quad \gamma = \beta - \frac{1}{2}\ln A. \tag{28}$$

We integrate (28) term by term by using the theorem of residues and the auxiliary formulas [10] for $a,b > 0$: formula No. 3.352.4

$$\int_0^{+\infty} \frac{e^{-bz} dz}{a+z} = -e^{ab}\,\mathrm{Ei}\,(-a\,b); \tag{29}$$

and formula No. 3.352.6

$$\mathrm{V.p.} \int_0^{+\infty} \frac{e^{-bz} dz}{a-z} = e^{-ab}\,\mathrm{Ei}\,(a\,b)$$

where Ei(z) is the exponential integral function.

The integral of the type

$$J_i = \int_C e^{i\omega t} (i\omega + s_i)^{-1} \ln(i\omega + s_i)\, d\omega,$$

presents some problems. It must be transformed to an integral along the contour C_i (Fig. 2) and calculated using Eq. (29) and formulas for the exponential integral function [10]: No. 8.214.1:

$$\mathrm{Ei}(z) = C + \ln(-z) + \sum_{n=1}^{\infty} z^n (n\, n!)^{-1}, \quad z < 0.$$

Then we obtain the expression $J_i = -2\pi \exp(-s_i t)\,(\ln t + C)$. After all calculations are done, we come to the formula

$$\varphi(t) = Q\lambda^{-1}(\varphi_1(t) + \varphi_2(t)),$$

$$\begin{gathered}
\varphi_1(t) = 2^{-1}b_0 t^{-1}\Big(\sum_{n=0}^{N-1}\exp(-s_n t) - \sum_{n=1}^{N}\exp(-y_n t)\Big) + \\
+ \sum_{n=o}^{N-1} a_n \exp(-s_n t)\Big(\gamma - \frac{1}{2}\sum_{i\neq n}\exp((s_i - s_n)t)\mathrm{Ei}((s_n - s_i)t) - \\
- \frac{1}{2}\sum_{i>n}\ln(s_i - s_n) + \frac{1}{2}\sum_{i=1}^{N}\exp((y_i - s_n)t)\mathrm{Ei}((s_n - y_i)t) + \\
+ \frac{1}{2}\sum_{i=n+1}^{N}\ln(y_i - s_n) + \frac{1}{2}C\Big),
\end{gathered} \tag{30}$$

$$\varphi_2(t) = \frac{1}{2}\sum_{n=0}^{N-1} a_n \exp(-s_n t)\ln t.$$

In (30) the term $\varphi_1(t)$ remains finite as t increases, while the term $\varphi_2(t)$ increases logarithmically (remember that, for $n > 0$, products of the type $s_n t$ are assumed finite!).

Thus, the function $\varphi_2(t)$ defines the asymptotic pressure attenuation curve in the relaxation theory of percolation. Formally, the effect of the relaxation processes can be considered by introducing a multiplier of the type $[1 + \sum a_i \cdot \exp(-s_i t)]$, where $a_i, s_i > 0$, into the classical formula for the pressure attenuation curve with a logarithmic asymptote.

REFERENCES

1. O. Yu. Dinariev, "Features of the pressure attenuation curve for relaxation percolation flow," Prikl. Mekh. Tekh. Fiz., No. 5, 106–111 (1991).
2. Yu. M. Molokovich, N. N. Nepreimerov, V. I. Pikuza, and A. V. Shtanin, Relaxation Percolation [in Russian], Izdatel'stvo Kazan'skogo Universiteta, Kazan' (1980).
3. Yu. M. Molokovich and P. P. Osipov, Fundamentals of the Theory of Relaxation Percolation [in Russian], Izdatel'stvo Kazan'skogo Universiteta, Kazan' (1987).
4. Yu. M. Molokovich, "Fundamentals of the theory of relaxation percolation," Problems in the Theory of Percolation and the Mechanics of Processes to Increase Petroleum Yield [in Russian], Nauka, Moscow (1987), pp. 142–153.
5. O. Yu. Dinariev and O. V. Nikolaev, "Relaxation phenomena in saturated porous media," Prikl. Matem. Mekh., No. 3, 469–475 (1989)
6. O. Yu. Dinariev and O. V. Nikolaev, "Generalizing Darcy's law for transient percolation conditions," Dokl. Akad. Nauk, **313**, No. 1, 31–36 (1990).
7. O. Yu. Dinariev, "The wave propagation velocity for transport processes with relaxation," Dokl. Akad. Nauk, **301**, No. 3, 1095–1097 (1988).
8. J. Stein and G. Weis, Introduction to Harmonic Analysis in Euclidean Space [Russian translation], Mir, Moscow (1974).
9. Bateman Manuscript Project, Higher Transcendental Functions. McGraw-Hill, New York (1953-55).
10. I. S. Gradshteyn and I. M.Ryzhik, Table of Integrals, Series, and Products, 4th ed. Academic Press, New York (1965).